

Derivation of the Born Rule from Operational Assumptions

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1 Introduction

Whence the Born rule? It is fundamental to quantum mechanics; it is the essential link between probability and a formalism which is otherwise deterministic; it encapsulates the measurement postulates. Gleason's theorem [4] is mathematically informative, but its premises are too strong to have any direct operational meaning: here the Born rule is derived more simply, from purely operational assumptions.

The argument we shall present is based on Deutsch's derivation of the Born rule from decision theory [2]. The latter was criticized by Barnum *et al* [1], but their objections hinged on ambiguities in Deutsch's notation that have recently been resolved by Wallace [12]; here we follow Wallace's formulation. The argument is not quite the same as Wallace's, however. Wallace draws heavily on the Everett interpretation, as well as on decision theory; like Deutsch, he is concerned with constraints on *subjective* probability, rather than any objective counterpart to it. In contrast, the derivation of the Born rule that we shall present is independent of decision theory, independent of the interpretation of probability, and independent of any assumptions about the measuring process. As such it applies to all the major foundational approaches to quantum mechanics.

We assume the conventional scheme for the description of experiments: an initial state, measured observable, and set of macroscopic outcomes. Given a description of this form, we assume there is a general algorithm for the expectation value of the observable outcomes (the Born rule is such an algorithm). The argument then takes the following form: for a particular class of experiments there are definite rules for determining such descriptions, based on simple operational rules, and theoretical assumptions that concern only the state-preparation device, not the measurement device. These rules imply that in general such experiments can be described in *different* ways. But the algorithm we are looking for concerns the expectation value of the observed outcomes, so applied to these different descriptions, it must yield the *same* expectation value. Constraints of this form are in fact sufficient to force the Born rule. *If* there is to be such an algorithm, then it is the Born rule.

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2 Multiple-Channel Experiments

The kinds of experiments we shall consider are limited in the following respects: they are repeatable; there is a clear distinction between the state preparation device and the detection and registration device; and - this the most important limitation - we assume that for a given state-preparation device, preparing the system to be measured in a definite initial state, the state can be resolved into *channels*, each of which can be independently blocked, in such a way that when only one channel is open the outcome of the experiment is *deterministic* - in the sense that if there is any outcome at all (on repetition of the experiment) it is always the *same* outcome. We further suppose that for every outcome there is at least one channel for which it is deterministic, and - in order to associate a definite initial state with a particular region of the apparatus - we suppose that all the channels are recombined prior to the measurement process proper.

For an example of such an experiment that measures spin, consider a neutron interferometer, where orthogonal states of spin (with respect to a given axis) are produced by a beam-splitter, each propagating along different arms of the interferometer, before being recombined prior to the measurement of that component of spin. For an example that measures position, consider an optical two-slit experiment, adapted so that the lensing system after the slits first brings the light into coincidence, but then focuses it on detectors in such a way that each can receive light from only one of the slits. It is not too hard to specify an analogous procedure in the case of momentum;² any number of familiar experiments can be converted into an experiment of this kind.

We introduce the following notation. Let there be d channels in all, with $D \leq d$ possible outcomes $u_j \in U$, $j = 1, \dots, D$. These outcomes are macroscopic events (e.g. positions of pointers). Let M denote the experiment that is performed when all the channels are open, and M_k , $k = 1, \dots, d$ the (deterministic) experiment that is performed when only the k^{th} channel is open. Let there be identifiable regions $r_1, r_2, \dots$ of the state-preparation device through which the system to be measured must pass (if it is to be subsequently detected at all - regardless of which channels are open). Call an experiment satisfying these specifications a *multiple-channel experiment*.

One could go further, and provide operational definitions of the initial states in each case, but we are looking for a probability algorithm that can be applied to states that are mathematically defined (so any operational definition of the initial state would eventually have to be converted into a mathematical one): we may as well work with the mathematical state from the beginning.

3 Models of Experiments

Turn now to the schematic, mathematical descriptions of experiments. Our assumptions are conventional: we suppose that an experiment is designed to

²The conventional method for preparing a beam of charged particles of definite momentum (by selecting for deflection in a magnetic field) can be adapted quite simply.

measure some observable $\hat{X}$ on a complex Hilbert space H , which for convenience we take to be of finite dimensionality;³ we suppose that the apparatus is prepared in some initial state ψ , normalized to one,⁴ and that on measurement one of a finite number of microscopic outcomes $\lambda_k \in Sp(\hat{X})$ results, $k = 1, \dots, d$ (we allow for repetitions, i.e. for some $j \neq k$ we may have $\lambda_j = \lambda_k$). We suppose that these microscopic events are amplified up to the macroscopic level by some physical process $\Omega : Sp(\hat{X}) \rightarrow U$, yielding one or other of the D possible displayed outcomes $u_j \in U$. We suppose the latter macroscopic events occur with probabilities p_j , $j = 1, \dots, D$.⁵

We take it that the *details* of the detection and amplification process are what are disputed, not that there is such a process, nor that it results in macroscopic outcomes u_j . The probabilities computed from records of repeated trials concern in the first instance these registered, macroscopic outcomes, not the unobservable microscopic events λ_k (indeed, on some approaches to foundations, there *are no* probabilistic microscopic events, prior to amplification up to the macroscopic level). To keep this distinction firmly in mind - and the distinction between the sets U and R - we shall not assume (as is usual) the numerical equality of $\Omega(\lambda_k)$ with λ_k ; we do, however, assume that the macroscopic outcomes $u_j \in U$ are physical numerals, so that addition and multiplication operations can be defined on them. For convenience we assume that none of these numerals is the zero.

Call the triple $\langle \psi, \hat{X}, \Omega \rangle$ an *experimental model*, denote g . This scheme extends without any modification to experiments where there are inefficiencies in the detection and registration devices, so long as they are the same for every channel. (A more sophisticated scheme will be needed if the efficiencies differ from one channel to the next, however; we neglect this complication here.)

This scheme applies to a much wider variety of experiments than multiple-channel experiments; the Born rule is conventionally stated in just these terms. We shall be interested in algorithms that assign real numbers to experimental models, interpreted as expectation values, i.e. weighted averages of the quantities u_j , with weights given by the probabilities p_j of each u_j , $j = 1, \dots, D$. We are therefore looking for a map $V : g \rightarrow R$ of the form:

$$V[\psi, \hat{X}, \Omega] = \sum_{j=1}^D p_j u_j, \quad \sum_{j=1}^D p_j = 1. \quad (1)$$

If $D = d$ we can write the u_j 's directly in terms of the $\Omega(\lambda_k)$'s. Otherwise define $\lambda^{-1}(u_j) = \{k : \Omega(\lambda_k) = u_j\}$, $j = 1, \dots, D$, and choose any real numbers

³It would be just as easy to work with Hilbert spaces of countably infinite dimension, and restrict instead the observables to self-adjoint operators with purely point spectra. (The difficulty with observables with continuous spectra is purely technical, however.)

⁴Later on we shall consider the consequences of relaxing the normalization condition (correspondingly, we use the term "state" loosely, to mean any Hilbert space vector defined up to phase).

⁵In the case of the Everett interpretation, we say rather that *all* of the macroscopic outcomes result, but that each of them is in a different branch (with a given amplitude). (We will consider the interpretation of probability in the Everett interpretation in due course.)

$w_k \in [0, 1]$, $k = 1, \dots, d$ such that $\sum_{k \in \lambda^{-1}(u_j)} w_k = p_j$. From Eq.(1) we obtain:

$$V[\psi, \hat{X}, \Omega] = \sum_{k=1}^d w_k \Omega(\lambda_k), \quad \sum_{k=1}^d w_k = 1. \quad (2)$$

Conversely, given any d real numbers $w_k \in [0, 1]$ satisfying Eq.(2), define the D numbers $p_j = \sum_{k \in \lambda^{-1}(u_j)} w_k$; from Eq.(2) we obtain Eq.(1).

In what follows, we assume the existence of probabilities p_j satisfying Eq.(1), *and therefore* that there are real numbers w_k satisfying Eq.(2). The latter will prove more convenient for calculations.

4 The Consistency Condition

Our general strategy is as follows. In the special case of multiple-channel experiments, there are clear criteria for when an experiment is to be assigned a given model. There follows an important constraint on V : for if M is assigned two *distinct* models g, g' , and if there is to be any general algorithm $V : g \rightarrow R$, then the expectation values it assigns to these two models had better agree, i.e. $V(g) = V(g')$. We view this as a *consistency condition* on V . Failing this condition, expectation values of models could have no unequivocal experimental meaning. The probabilistic outcome events $u_k \in U$ that we are talking of are all observable; it is the mean values of these that the quantities $V(g)$ concern; if one and the same mean value is matched to two expectation values, $V(g) \neq V(g')$, then either the experiment cannot be modelled by g and g' , or *there is no algorithm* V for mapping models to expectation values.

That a condition of this kind played a tacit role in Deutsch's derivation was recognized by Wallace; it was used explicitly in Wallace's deduction [12] of the Born rule, although there it was cast in a slightly different form, and the conditions for its use were stated in terms of the Everett theory of measurement (including the theory of the detection and registration process). Here we make do with operational criteria, and with assumptions about the behavior of the state *prior* to any detection events; we suppose that this prior evolution of the state is purely deterministic, and governed by the unitary formalism of quantum mechanics.⁶

Consider a multiple-channel experiment M . By assumption, there are d deterministic experiments M_k , $k = 1, \dots, d$ that can also be performed with this apparatus, on blocking every channel save the k^{th} , each yielding one of the D macroscopic outcomes $u_j \in U$. Given that the initial state in region r for M_k is φ_k , it is clear enough, on operational grounds, as to what can be counted as a model for this experiment: the experiment measures any $\hat{X}$ such that $\hat{X}\varphi_k = \lambda_k\varphi_k$, for any λ_k and any Ω such that $\Omega(\lambda_k) \in U$ is the outcome of M_k .

⁶Of course in its initial phases the process of state preparation will involve probabilistic events, if only in collimating particles produced from the source, or in blocking particular channels. But it does not matter what these probabilities are; all that matters is that *if* a particle is located in a given region of the apparatus, *then* it is in a definite state, and unitarily develops in a definite way (prior to any detection or registration process).

Now consider the indeterministic experiment M with every channel open. We suppose that the state of M at r is $\psi = \sum_{k=1}^d c_k \varphi_k$; then the observable measured is any $\hat{X}$ such that $\hat{X}\varphi_k = \lambda_k \varphi_k$ for $k = 1, \dots, d$, and any Ω such that $\Omega(\lambda_k) \in U$ is the outcome of each M_k .

Let us state this as a definition:

Definition 1 *Let M have d channels and D outcomes. Then M realizes $\langle \psi, \hat{X}, \Omega \rangle$ if and only if*

- (i) *for some region r and orthogonal states $\{\varphi_k\}$, φ_k is the state of M_k in r , $k = 1, \dots, d \geq D$, and $\psi = \sum_{k=1}^d c_k \varphi_k$ is the state of M in r ,*
- (ii) $\hat{X}\varphi_k = \lambda_k \varphi_k$, $k = 1, \dots, d$,
- (iii) $\Omega(\lambda_k)$ is the outcome of M_k , $k = 1, \dots, d$.

The definition applies equally to a deterministic experiment (the limiting case in which $d = D = 1$). Bearing in mind that from our definition of multiple-channel experiments, for each $u_j \in U$, there is at least one M_k for which u_j is deterministic, it follows from (ii), (iii) that $\hat{X}$ has at least D distinct eigenvalues.

Why is it right to model experiments in this way and not some other? The deterministic case speaks for itself; in the indeterministic case, the short answer is that it is underwritten by the linearity of the equations of motion. An apparatus that *deterministically* measures each eigenvalue λ_k of $\hat{X}$, when the state in a given region of the apparatus is φ_k , will *indeterministically* measure the eigenvalues λ_k of $\hat{X}$, when the state in that region is in a superposition of the φ_k 's. This principle is implicit in standard laboratory procedures; this is how measuring devices are standardly calibrated, and how their functioning is checked.

The consistency condition now reads:

Definition 2 *V is consistent if and only if $V(g) = V(g')$ whenever g and g' can be realized by the same experiment.*

In the deterministic case evidently:

$$V[\varphi_k, \lambda_k \hat{P}_{\varphi_k}, \Omega] = \Omega(\lambda_k). \quad (3)$$

We will show that if $|\psi| = 1$ and V is consistent, with $\langle, \rangle$ the inner product on H , then⁷

$$V[\psi, \hat{X}, \Omega] = \langle \psi, \Omega(\hat{X})\psi \rangle \quad (4)$$

Eq.(4) is the Born rule.

We begin with some simple consequences of the consistency condition. The Born rule is then derived in stages: first for equal norms in the simplest possible

⁷Whilst $\Omega(\hat{X})$ makes no sense as an operator (as the values of Ω are physical numerals like pointer-positions, not real numbers) we are assuming that arithmetic operations can be defined for the $\Omega(\lambda_k)$'s; define $\langle \psi, \Omega(\lambda_k) \hat{P}_{\varphi_k} \psi \rangle = \Omega(\lambda_k) \langle \psi, \hat{P}_{\varphi_k} \psi \rangle$ accordingly, and extend by linearity.

case of a spin half system; then for the general case of equal norms; and then for rational norms. The general case of irrational norms is handled by a simple continuity condition. As promised, we shall also derive a probability rule for initial states normalized to arbitrary finite numbers.

5 Consequences of the Consistency Condition

We prove four general constraints on V that follow from consistency. (Eqs.(5)-(8) may be found in Wallace [12], derived on somewhat different assumptions.) In each case an equality is derived from the fact that a single experiment realizes two *different* models: by consistency, each must be assigned the same expectation value.

We assume it is not in doubt that there do exist such experiments, in which the initial state (prior to any detection or amplification process) evolves unitarily in the manner stated.

Lemma 3 *Let V be consistent. It follows*

(i) for invertible $f : R \rightarrow R$:

$$V[\psi, \hat{X}, \Omega] = V[\psi, f(\hat{X}), \Omega \circ f^{-1}]. \quad (5)$$

(ii) For orthogonal projectors $\{\hat{P}_k\}$, $k = 1, \dots, d$, such that $\hat{P}_k \varphi_j = \delta_{kj} \varphi_j$

$$V\left[\sum_{k=1}^d c_k \varphi_k, \sum_{k=1}^d \lambda_k \hat{P}_{\varphi_k}, \Omega\right] = V\left[\sum_{k=1}^d c_k \varphi_k, \sum_{k=1}^d \lambda_k \hat{P}_k, \Omega\right]. \quad (6)$$

(iii) For $\hat{U}_\theta : \varphi_k \rightarrow e^{i\theta_k} \varphi_k$, $k = 1, \dots, d$, for arbitrary $\theta_k \in [0, 2\pi] \subset R$

$$V\left[\psi, \sum_{k=1}^d \lambda_k \hat{P}_{\varphi_k}, \Omega\right] = V\left[\hat{U}_\theta \psi, \sum_{k=1}^d \lambda_k \hat{P}_{\varphi_k}, \Omega\right]. \quad (7)$$

(iv) For $\hat{U}_\pi : \varphi_k \rightarrow \varphi_{\pi(k)}$, where π is any permutation of $\{1, \dots, d\}$

$$V[\psi, \hat{X}, \Omega] = V[\hat{U}_\pi \psi, \pi^{-1}(\hat{X}), \Omega]. \quad (8)$$

Proof. Let $g = \langle \psi, \hat{X}, \Omega \rangle$ be realized by M with d channels. Then for some region r_1 the state of M_k is φ_k , $k = 1, \dots, d$, that of M is $\sum_{k=1}^d c_k \varphi_k$, and there exist (not necessarily distinct) real numbers $\lambda_1, \dots, \lambda_k$ such that $\hat{X} \varphi_k = \lambda_k \varphi_k$, $\Omega(\{\lambda_k\}) = U$. Since for invertible f , $\Omega[f^{-1}(f(\lambda_k))] = \Omega(\lambda_k)$, $f(\hat{X}) \varphi_k = f(\lambda_k) \varphi_k$, M realizes $\langle \psi, f(\hat{X}), \Omega \circ f^{-1} \rangle$, and (i) follows from consistency. Further, M realizes any other model $\langle \psi, \hat{Y}, \Omega \rangle$ such that $\hat{Y} \varphi_k = \lambda_k \varphi_k$; $\sum_{k=1}^d \lambda_k \hat{P}_k$ is such a $\hat{Y}$, so (ii) follows from consistency. Suppose now that ψ evolves unitarily to

the state $\widehat{U}_\theta \psi$ in region r_2 . Then in r_2 the state of each M_k is $e^{i\theta_k} \varphi_k$, and since $\widehat{P}_{e^{i\theta_k} \varphi_k} = \widehat{P}_{\varphi_k}$, M realizes $\langle \widehat{U}_\theta \psi, \sum_{k=1}^d \lambda_k \widehat{P}_{\varphi_k}, \Omega \rangle$, and (iii) follows from consistency. Finally, let ψ subsequently evolve to the state $\widehat{U}_\pi \psi$ in region r_3 . Then in r_3 the state of each M_k is $\varphi_{\pi(k)}$, and the state of M is $\sum_{k=1}^d c_k \varphi_{\pi(k)}$. Without loss of generality, we may write $\widehat{X}$ as $\sum_{k=1}^d \lambda_k \widehat{P}_{\varphi_k}$; then $\pi^{-1}(\widehat{X}) = \sum_{k=1}^d \lambda_k \widehat{P}_{\varphi_{\pi(k)}}$ satisfies $\pi^{-1}(\widehat{X}) \varphi_{\pi(k)} = \lambda_k \varphi_{\pi(k)}$, so M realizes $\langle \widehat{U}_\pi \psi, \pi^{-1}(\widehat{X}), \Omega \rangle$, and (iv) follows from consistency ■

Eqs.(5)-(8) are of course trivial consequences of the Born rule, Eq.(4). Note further that in each case the observables whose expectation values are identified *commute* - these are constraints among probability assignments to projectors belonging to a *single* resolution of the identity. Finally, note that the normalization of the initial state ψ played no role in the proofs.

6 Case 1: The Stern-Gerlach Experiment for Equal Norms

Consider the Stern-Gerlach experiment with $d = D = 2$. Let $\widehat{X} = \frac{1}{2} \widehat{P}_+ - \frac{1}{2} \widehat{P}_- = \widehat{\sigma}_z$ (in conventional notation), the observable for the z -component of spin with eigenstates $\varphi_\pm$, and let $\psi = c_+ \varphi_+ + c_- \varphi_-$. Let $\widehat{U}_\pi$ interchange φ_+ and φ_- , so $\widehat{U}_\pi \widehat{\sigma}_z \widehat{U}_\pi^{-1} = -\widehat{\sigma}_z$. From Lemma 3(iv) it follows that:

$$V[c_+ \varphi_+ + c_- \varphi_-, \widehat{\sigma}_z, \Omega] = V[c_+ \varphi_- + c_- \varphi_+, -\widehat{\sigma}_z, \Omega]. \quad (9)$$

From Eq.(9) and Lemma 3(i):

$$V[c_+ \varphi_+ + c_- \varphi_-, \widehat{\sigma}_z, \Omega] = V[c_+ \varphi_- + c_- \varphi_+, \widehat{\sigma}_z, \Omega \circ -I] \quad (10)$$

(where $(\Omega \circ -I)(x) = \Omega(-x)$). From Eq.(10), in the special case that $|c_+|^2 = |c_-|^2$, and using Lemma 3(iii) to compensate for any differences in phase:

$$V[c_+ \varphi_+ + c_- \varphi_-, \widehat{\sigma}_z, \Omega] = V[c_+ \varphi_+ + c_- \varphi_-, \widehat{\sigma}_z, \Omega \circ -I]. \quad (11)$$

Consider the LHS of this equality. From Eq.(2), writing $w_1 = w$, $w_2 = 1 - w$, $\Omega(\pm \frac{1}{2}) = \Omega(\pm)$ - so that $\Omega(+)$ results with probability w , and $\Omega(-)$ results with probability $1 - w$ - we obtain the expectation value $x = w\Omega(+) + (1 - w)\Omega(-)$. But by similar reasoning, the RHS yields $w\Omega(-) + (1 - w)\Omega(+)$ - so $x = -w\Omega(+) + w\Omega(-)$. Equating the two, $x = \frac{1}{2}[\Omega(+) + \Omega(-)]$.

We have shown, for $|c_+|^2 = |c_-|^2$:

$$V[c_+ \varphi_+ + c_- \varphi_-, \widehat{\sigma}_z, \Omega] = \frac{1}{2} \Omega(+) + \frac{1}{2} \Omega(-) \quad (12)$$

in accordance with the Born rule. Note that here we have derived an expectation values in a situation (dimension 2) where Gleason's theorem does not apply. (Note that the normalization of the initial state ψ is again irrelevant to the result.)

7 Case 2: General Superpositions of Equal Norms

Consider an arbitrary observable on any d -dimensional subspace H_d of Hilbert space. By the spectral theorem, we may write $\hat{X} = \sum_{k=1}^d \lambda_k \hat{P}_{\varphi_k}$, for some set of orthogonal vectors $\{\varphi_k\}$, $k = 1, \dots, d$ spanning H_d , where there may be repetitions among the λ_k 's. Let ψ be a (not-necessarily normalized) vector in H_d ; then for some d -tuple of complex numbers $\langle c_1, \dots, c_d \rangle$, $\psi = \sum_{k=1}^d c_k \varphi_k$. For any permutation π , we have from Lemma 3(iv), (i):

$$V[\sum_{k=1}^d c_k \varphi_k, \hat{X}, \Omega] = V[\sum_{k=1}^d c_k \varphi_{\pi(k)}, \pi^{-1}(\hat{X}), \Omega] = V[\sum_{k=1}^d c_k \varphi_{\pi(k)}, \hat{X}, \Omega \circ \pi]. \quad (13)$$

If $|c_k|^2 = |c_j|^2$, $j, k = 1, \dots, d$, using Lemma 3(iii) as before to adjust for any phase differences

$$V[\psi, \hat{X}, \Omega] = V[\psi, \hat{X}, \Omega \circ \pi]. \quad (14)$$

Let $\langle w_1, \dots, w_d \rangle$ be a d -tuple of non-negative real numbers satisfying Eq.(2). From Eq.(14):

$$\sum_{k=1}^d w_k \Omega(\lambda_k) = \sum_{k=1}^d w_k \Omega(\lambda_{\pi(k)}). \quad (15)$$

Eq.(15) holds for any permutation; let π interchange j and k , and otherwise act as the identity. There follows

$$w_j \Omega(\lambda_j) + w_k \Omega(\lambda_k) = w_k \Omega(\lambda_j) + w_j \Omega(\lambda_k). \quad (16)$$

Conclude that if $\Omega(\lambda_j) \neq \Omega(\lambda_k)$ then $w_k = w_j$ (recall that by convention $0 \notin U$, so $\Omega(\lambda_k)$ is never zero).

If $D = d$, evidently $w_k = w_j$ for all $j, k = 1, \dots, d$. Since $\sum_k^d w_k = 1$, $w_k = \frac{1}{d}$, $k = 1, \dots, d$. Therefore

$$V[\psi, \hat{X}, \Omega] = \frac{1}{d} [\sum_{k=1}^d \Omega(\lambda_k)]. \quad (17)$$

If not, suppose $\Omega(\lambda_j) = \Omega(\lambda_k)$ for $j, k = 1, \dots, b < d$. (If $b = d$ Eq.(17) follows trivially.) For any j, k such that $b < j \leq d$, $k \leq b$, $\Omega(\lambda_k) \neq \Omega(\lambda_j)$, from which we conclude as before that $w_k = w_j$. Note further that under the stated conditions, $1/d = |c_k|^2 (\sum_{j=1}^d |c_j|^2)^{-1}$. We have proved

Theorem 4 *Let $\psi = \sum_{k=1}^d c_k \varphi_k$, where $|c_k|^2 = |c_j|^2$ for all $j, k = 1, \dots, d$. Then if V is consistent*

$$V[\sum_{k=1}^d c_k \varphi_k, \sum_{k=1}^d \lambda_k \Omega_{\varphi_k}, \Omega] = \sum_{k=1}^d \frac{|c_k|^2}{\sum_{j=1}^d |c_j|^2} \Omega(\lambda_k). \quad (18)$$

Like Lemma 3, Theorem 4 is independent of the normalization of ψ .

8 Case 3: d=2 Normalized Superpositions with Rational Norms

The idea for extending these methods to treat the case of unequal but rational norms is as follows: consider an experiment in which the initial state ψ evolves deterministically so that each component φ_k entering into the initial superposition with amplitude c_k evolves into a superposition of z_k orthogonal states of equal norm $1/\sqrt{z_k}$, such that $|c_k/\sqrt{z_k}|^2$ is constant for all k . One can then show that the experiment has a model in which the initial state is a superposition of states of equal norms, so Theorem 4 can be applied. (Evidently for this to work each $|c_k|^2$ will have to be a rational number.)

For simplicity, consider first the case $d = 2$ for real amplitudes. Let $\psi = \frac{\sqrt{m}}{\sqrt{m+n}}\varphi_1 + \frac{\sqrt{n}}{\sqrt{m+n}}\varphi_2$, where m and n are integers. Let $\hat{X} = \lambda_1\hat{P}_{\varphi_1} + \lambda_2\hat{P}_{\varphi_2}$. We will show that if V is consistent, $V[\psi, \hat{X}, \Omega] = \frac{m}{m+n}\Omega(\lambda_1) + \frac{n}{m+n}\Omega(\lambda_2)$. Let the deterministic experiments of M be M_1, M_2 , with registered outcomes $\Omega(\lambda_1), \Omega(\lambda_2)$ respectively. Let the initial states of M, M_1, M_2 in region r_1 be $\psi, \varphi_1, \varphi_2$ respectively. Then M realizes $g_1 = \langle \psi, \hat{X}, \Omega \rangle$. Now let ψ evolve to $\hat{U}\psi$ in region r_2 , where $\hat{U}\varphi_1 = \frac{1}{\sqrt{m}}\sum_{k=1}^m\chi_k$, $\hat{U}\varphi_2 = \frac{1}{\sqrt{n}}\sum_{k=m+1}^{n+m}\chi_k$, for some orthogonal set of vectors $\{\chi_k\}$, $k = 1, \dots, m+n$. Denote $\lambda_1\hat{P}_{\hat{U}\varphi_1} + \lambda_2\hat{P}_{\hat{U}\varphi_2}$ by $\hat{Y}$. Then the initial state of M_i , $i = 1, 2$ is $\hat{U}\varphi_i$ in r_2 , whilst that of M is $c_1\hat{U}\varphi_1 + c_2\hat{U}\varphi_2$ in r_2 ; since $\hat{Y}\hat{U}\varphi_i = \lambda_i\hat{U}\varphi_i$ it follows that M realizes $g_2 = \langle \hat{U}\psi, \hat{Y}, \Omega \rangle$. By consistency, $V(g_1) = V(g_2)$. Now define $\hat{P}_1 = \lambda_1\sum_{k=1}^m\hat{P}_{\chi_k}$, $\hat{P}_2 = \lambda_2\sum_{k=m+1}^{n+m}\hat{P}_{\chi_k}$; since $\hat{P}_k\hat{U}\varphi_j = \delta_{kj}\hat{U}\varphi_j$, $k, j = 1, 2$, by Lemma 3(ii) it follows $V[\hat{U}\psi, \hat{Y}, \Omega] = V[\hat{U}\psi, \lambda_1\hat{P}_1 + \lambda_2\hat{P}_2, \Omega]$. But $\hat{U}\psi = \frac{1}{\sqrt{m+n}}\sum_{k=1}^{n+m}\chi_k$; applying Theorem 4 for $d = m+n$, and noting that $\Omega(\lambda_k) = \lambda_1$ for $k = 1, \dots, m$, and λ_2 otherwise, the result follows.

9 Case 4: General Superpositions with Rational Norms

The argument just given assumed ψ was normalized to one. The standard rationale for this is of course based on the probabilistic interpretation of the state, and hence, at least tacitly, on the Born rule. It may be objected that we are only able to derive the dependence of the expectation value on the *squares* of the norms of the initial state, because this is put in by hand from the beginning. But this suspicion is unfounded. Suppose, indeed, only that $\frac{|c_1|^2}{|c_2|^2} = \frac{m}{n}$. As before, define $\hat{U}\varphi_1 = \frac{1}{\sqrt{m}}\sum_{k=1}^m\chi_k$, $\hat{U}\varphi_2 = \frac{1}{\sqrt{n}}\sum_{k=m+1}^{n+m}\chi_k$. The state $\hat{U}\psi$ in region r_2 will have whatever normalization ψ had in r_1 ; the states $\hat{U}\varphi_i$ $i = 1, 2$ will be eigenstates of $\hat{P}_i$, as before; Definition 1,2 will apply as before. Conclude that if V is consistent, $V[\psi, \hat{X}, \Omega] = V[\hat{U}\psi, \lambda_1\hat{P}_1 + \lambda_2\hat{P}_2, \Omega]$, as before. The difference is

that now $\hat{U}\psi = \frac{c_1}{\sqrt{m}} \sum_{k=1}^m \chi_k + \frac{c_2}{\sqrt{n}} \sum_{k=m}^{n+m} \chi_k = \frac{c_1}{\sqrt{m}} \sum_{k=1}^{n+m} \chi_k = \frac{c_2}{\sqrt{n}} \sum_{k=1}^{n+m} \chi_k$ (adjusting the phases of c_1 and c_2 , using Lemma 3(ii), as required). Evidently we have an initial state which is a superposition of $n+m$ components of equal norm, m of which yield outcome $\Omega(\lambda_1)$ and n of which yield outcome $\Omega(\lambda_2)$. Since $\frac{m}{n+m} = \frac{|c_1|^2}{|c_1|^2 + |c_2|^2}$, $\frac{n}{n+m} = \frac{|c_2|^2}{|c_1|^2 + |c_2|^2}$

$$V[\psi, \lambda_1 \hat{P}_{\varphi_1} + \lambda_2 \hat{P}_{\varphi_2}, \Omega] = \frac{|c_1|^2}{|c_1|^2 + |c_2|^2} [\Omega(\lambda_1)] + \frac{|c_2|^2}{|c_1|^2 + |c_2|^2} [\Omega(\lambda_2)]. \quad (19)$$

Evidently the normalization of ψ is irrelevant.

This result is worth proving in full generality:

Theorem 5 *For each $i, j = 1, \dots, d$ let $c_i \in C$ satisfy $|c_i| > 0$, $\frac{|c_i|^2}{|c_j|^2} \in Z$. Then*

$$V[\sum_{k=1}^d c_k \varphi_k, \sum_{k=1}^d \lambda_k \hat{P}_{\varphi_k}, \Omega] = \sum_{k=1}^d \frac{|c_k|^2}{\sum_{j=1}^d |c_j|^2} \Omega(\lambda_k). \quad (20)$$

Proof. For $\{c_k\}$ as stated, there exists $c \in C$, $z_k \in Z$, $\theta_k \in [0, 2\pi]$, $k = 1, \dots, d$ such that $c_k = ce^{i\theta_k} \sqrt{z_k}$. Let m_k, n be integers such that $z_k = \frac{m_k}{n}$, $k = 1, \dots, d$; let $\{\chi_j\}$, $j = 1, \dots, s$ be an orthonormal basis on an s -dimensional subspace of Hilbert space H_s , where $s = \sum_{k=1}^d m_k$ (we may suppose for $j = 1, \dots, d$, $\chi_j = \varphi_j$). Define $\hat{U}$ on H^s by the action $\hat{U}\varphi_k = \frac{1}{\sqrt{m_k}} \sum_{j=m_k+1}^{m_k+1} \chi_j$; let $\hat{P}_k = \hat{P}_{\hat{U}\varphi_k}$, $k = 1, \dots, d$. Let $\psi = \sum_{k=1}^d c_k \varphi_k$; let M realize $g_1 = \langle \psi, \sum_{k=1}^d \lambda_k \hat{P}_{\varphi_k}, \Omega \rangle$; then for some region r_1 , the initial state of M is ψ and the state of each M_k is φ_k with outcome $\Omega(\lambda_k)$. Let the state of M at r_2 be $\hat{U}\psi$; then M also realizes $g_2 = \langle \hat{U}\psi, \sum_{k=1}^d \lambda_k \hat{P}_k, \Omega \rangle$, and by consistency $V(g_1) = V(g_2)$. But by construction

$$\hat{U}\psi = \sum_{k=1}^d c_k \hat{U}\varphi_k = \sum_{k=1}^d \frac{c_k}{\sqrt{m_k}} \sum_{j=m_k+1}^{m_k+1} \chi_j = \sum_{k=1}^d \frac{ce^{i\theta_k}}{n} \sum_{j=m_k+1}^{m_k+1} \chi_j \quad (21)$$

so $V[\hat{U}\psi, \sum_{k=1}^d \lambda_k \sum_{j=m_k+1}^{m_k+1} \hat{P}_{\chi_j}, \Omega] = V[\frac{c}{n} \sum_{k=1}^s \chi_s, \sum_{k=1}^d \lambda_k \sum_{j=m_k+1}^{m_k+1} \hat{P}_{\chi_s}, \Omega]$ (by Lemma 3(ii)). The result follows from Theorem 4 (of s equiprobable outcomes, m_k have outcome $\Omega(\lambda_k)$, so $V(g) = \frac{1}{s} \sum_{k=1}^d m_k \Omega(\lambda_k)$. But $m_k/s = m_k(\sum_{j=1}^d m_j)^{-1} = z_k(\sum_{j=1}^d z_j)^{-1} = |c_k|^2 \sum_{j=1}^d |c_j|^2)^{-1}$) ■

Examination of the proof shows that the dependence of probabilities on the modulus square of the expansion coefficients of the state ultimately derives from the fact that we are concerned with unitary evolutions on Hilbert space, specifically an *inner-product* space, and not some general normed linear topological space. A general class of norms on the latter is of the form $\|x\| = \left(\sum_{k=1}^d |\xi_k|^p\right)^{1/p}$, $1 \leq p < \infty$ (d may also be taken as infinite). Such spaces (l^p spaces) are metric spaces and can be completed in norm. The proof

as we have developed it would apply equally to a theory of unitary (i.e. invertible norm-preserving) motions on such a space, yielding the probability rule

$$V[\sum_{k=1}^d c_k \varphi_k, \sum_{k=1}^d \lambda_k \hat{P}_{\varphi_k}, \Omega] = \sum_{k=1}^d \frac{|c_k|^p}{\sum_{j=1}^d |c_j|^p} \Omega(\lambda_k) \quad (22)$$

(assuming that $\frac{|c_k|^p}{\sum_{j=1}^d |c_j|^p} \in Z$, $j, k = 1, \dots, d$). But the only space of this form which is an inner-product space is $p = 2$ (Hilbert space).

10 Case 5: Arbitrary States

There are a variety of possible strategies for the treatment of irrational norms, but the one that is most natural, given that we are making use of operational criteria for the interpretation of experiments, is to weaken these criteria in the light of the limitations of *realistic* experiments. In practise, one would not expect *precisely* the same state to be prepared on each run of the experiment. Properly speaking, the statistics actually obtained will be those for an *ensemble* of experiments; correspondingly, they should be obtained from a family of models, differing slightly in their initial states. We should therefore speak of *approximate* models (or of models that are *approximately* realized) - where the differences among the models are small.

How small is small? What is the topology on the space of states? The obvious answer, from a theoretical point of view, is the norm topology. We should suppose that for sufficiently small ϵ , so long as $|\psi - \psi'| < \epsilon$, then if $\langle \psi, \hat{X}, \Omega \rangle$ is an approximate model for M then so is $\langle \psi', \hat{X}, \Omega \rangle$. Indeed, $\hat{X}$ and Ω will likewise be subject to small variations. (Only the outcome set U can be regarded as precisely specified, insofar as outcomes are identified with numerals.)

But now it is clear that the details are hardly important; any algorithm that applies to families of models of this type, yielding expectation values, will have to be *continuous* in the norm topology. Given that, the extension of Theorem 5 to the irrational case is trivial. We define:

Definition 6 Let $g^{(i)}$ be any sequence of models $\langle \psi^{(i)}, \hat{X}, \Omega \rangle$, $i = 1, 2, \dots$ such that $\lim_{i \rightarrow \infty} |\psi^{(i)} - \psi| = 0$. Then V is continuous in norm if $\lim_{i \rightarrow \infty} V(g^{(i)}) = V(g)$.

We may finally prove:

Theorem 7 Let V be consistent and continuous in norm. Then for any model $\langle \psi, \hat{X}, \Omega \rangle$

$$V[\psi, \hat{X}, \Omega] = \frac{\langle \psi, \Omega(\hat{X})\psi \rangle}{\langle \psi, \psi \rangle}. \quad (23)$$

Proof. It is enough to prove that any realizable model satisfies Eq.(22). If realizable, there is some multiple-channel experiment M with d channels and D outcomes that realizes $\langle \psi, \hat{X}, \Omega \rangle$. Let $\{\varphi_k\}$, $k = 1, \dots, d$ be any orthogonal family of vectors such that $\hat{X}\varphi_k = \lambda_k\varphi_k$ (not all the λ_k 's need be distinct). Without loss of generality, let $\psi = \sum_{k=1}^d c_k\varphi_k$, $\hat{X} = \sum_{k=1}^d \lambda_k\hat{P}_{\varphi_k}$. Let $0 < \epsilon = \sum_{k=1}^d |c_k|^2$ and let $\{c_k^{(i)}\} \subseteq C^d$ be any sequence of d -tuples such that $\epsilon \leq \sum_{k=1}^d |c_k^{(i)}|^2$, $\frac{|c_j^{(i)}|^2}{|c_k^{(i)}|^2} \in Z$, $\lim_{i \rightarrow \infty} c_k^{(i)} = c_k$ (such a sequence can always be found). Let $\psi^{(i)} = \sum_{k=1}^d c_k^{(i)}\varphi_k$, $g^{(i)} = \langle \psi^{(i)}, \hat{X}, \Omega \rangle$. By Theorem 5, $V[\psi^{(i)}, \hat{X}, \Omega] = \sum_{k=1}^d \frac{|c_k^{(i)}|^2}{\sum_{j=1}^d |c_j^{(i)}|^2} \Omega(\lambda_k) = \frac{1}{\sum_{j=1}^d |c_j^{(i)}|^2} \sum_{k=1}^d \Omega(\lambda_k) \langle \psi^{(i)}, \hat{P}_{\varphi_k} \psi^{(i)} \rangle$. The numerator is $\langle \psi^{(i)}, \sum_{k=1}^d \Omega(\lambda_k) \hat{P}_{\varphi_k} \psi^{(i)} \rangle$ (by the continuity of the inner product), i.e. $\langle \psi^{(i)}, \Omega(\hat{X}) \psi^{(i)} \rangle$; since the denominator is bounded below by $\epsilon > 0$, with $\lim_{i \rightarrow \infty} \sum_{j=1}^d |c_j^{(i)}|^2 = \sum_{j=1}^d |c_j|^2$, and since $\lim_{i \rightarrow \infty} \langle \psi^{(i)}, \Omega(\hat{X}) \psi^{(i)} \rangle = \langle \psi, \Omega(\hat{X}) \psi \rangle$ (again by the continuity of the inner product), the result follows from the continuity of V ■

A similar proof can be given for a general probability rule on l^p spaces, $p \neq 2$ (i.e. Eq.(22), for arbitrary complex coefficients; of course this result could not be expressed as in Eq.(23), using an inner product).

Is a continuity assumption permitted in the present context? Gleason's theorem does not require it; if one is going to do better than Gleason's theorem, it would be pleasant to derive the continuity of the probability measure, rather than to assume it. But from an operational point of view continuity is a very natural assumption: no algorithm that could ever be used is going to distinguish between states that differ infinitesimally.

11 A Role for Decision Theory

Deutsch [2] took a rather different view: he was at pains to establish the Born rule for irrational norms, without assuming continuity. His method, however, was far from operational: along with axioms of decision theory, he assumed that quantum mechanics is *true* (under the Everett interpretation).

A hybrid is possible: the present method can in fact be supplemented with axioms of decision theory, yielding the Born rule for irrational norms, without any continuity assumption. But as Wallace [12] makes clear, nothing much hangs on this question. One can do without a continuity assumption, but there are just as good reasons to invoke it from a decision theoretic point of view as from an operational one. In neither case is there any reason to distinguish between states that differ infinitesimally.

Decision theory is important for a rather different reason: it is because the non-probabilistic parts of decision theory (as Deutsch puts it), or decision theory in the face of uncertainty (as Wallace puts it) *can provide an account of*

probability in terms of something else. This matters in the case of the Everett interpretation; according to many, the Everett interpretation has no place for probability [7]; given Everett, probability cannot be taken as *primitive*.

So it is clear why Deutsch took the more austere line: if Everett is to be believed, quantum mechanics *is* purely deterministic. Deutsch supposed that the fundamental concept (that can be taken as primitive) is rather the *value* or the *utility* that an agent places upon a model - that $V(g)$ is in fact a utility. He argued that experiments should be thought of as games; for each registered outcome in U , we are to associate some utility, fixed in advance. So, in effect, the mapping $\Omega : \lambda_k \rightarrow \Omega(\lambda_k) \in U$ defines the *payoff* for the outcome λ_k .

Decision theory on this approach has a substantial role. If we suppose that the utilities of a rational agent are ordered, and satisfy very general assumptions ("axioms of rationality"), a representation theorem can be derived [10] which *defines* subjective probability in terms of the ordering of an agent's utilities. In effect, one deduces - in accordance with these axioms - that the agent acts *as if* she places such-and-such subjective probabilities on the outcomes of various actions.⁸

It is important that one can still make sense of uncertainty in this context, as Wallace explains. It may be we cannot help ourselves to probabilistic ideas *ab initio*, but that does not mean that one only deals with certainties - that games, in some sense, have only a *single* payoff, as Deutsch at one point suggests [2, p.3132-3.] From a first-person perspective, one does not know what outcome of a quantum game to expect to observe (there is certainly no first-person perspective from which they can *all* be observed). In fact, it is enough that - in the face of branching - a rational agent expects anything at all (that she does not expect *oblivion* [9]).

On this line of thought, the proofs of the Born rule just presented make an illegitimate assumption: Eq.(1). We are not entitled to assume that the macroscopic outcomes $u_j \in U$, $j = 1, \dots, D$ occur with probabilities p_j , for they all occur; so neither can we assume there are non-negative real numbers, summing to one, satisfying Eq.(2). But the proof of Theorem 4 (hence 5 and 7) depended on this assumption. Of course we may, with Deutsch and Wallace, eventually be in a position to make statements about the subjective probabilities of branches, but if so such statements will have to come at a later stage - *after* establishing the values $V(g)$ of various games. But then how are we to establish these values?

Here Wallace has provided a considerably more detailed analysis than Deutsch, and from weaker premises. But the proofs are correspondingly more compli-

⁸This does not mean that subjective probabilities are illusory, and correspond to nothing in reality. The point is to *legitimate* the concept, not to abolish it. As for its objective correlate, the most popular candidate has long been relative frequency (of outcomes in a sequence of trials). Relative frequencies are obviously important when it comes to *evidence* for probabilities, but there are well-known difficulties with trying to *identify them* with probabilities (for anything short of infinite sequences). We read Everett as making a contrary proposal: that the objective correlates of subjective probability are branches in the universal state (with respect to the decoherence basis). Here we are deducing the quantitative rule to be used in assigning subjective probabilities to branches.

cated; for the sake of simplicity we shall only consider Deutsch's argument, removing the ambiguities of notation in the way shown by Wallace.

First, consider Case 1, the Stern-Gerlach experiment. All is in order up to Eq.(11), but we must do without the assumption subsequently made - that the registered outcome $\Omega(+)$ results with probability w , and outcome $\Omega(-)$ with probability $1 - w$. Here Deutsch invokes a new principle, what he calls *the zero-sum rule*:

$$V[\varphi, \hat{X}, \Omega] = -V[\varphi, \hat{X}, -\Omega]. \quad (24)$$

Following Deutsch, let us assume that the numerical value of the utility $\Omega(\lambda_k)$ equals λ_k . Then, in the special case where $\lambda_1 = -\lambda_2$ (true for the measurement of a component of spin), from Eq.(24), applied to Eq.(11), we deduce:

$$V[c_1\varphi_1 + c_2\varphi_2, \hat{\sigma}_z, \Omega] = -V[c_1\varphi_1 + c_2\varphi_2, \hat{\sigma}_z, \Omega] \quad (25)$$

and hence that $V[c_1\varphi_1 + c_2\varphi_2, \hat{\sigma}_z, \Omega] = 0$, in accordance with the Born rule in this special case.

Although evidently of limited generality, the result is illustrative - assuming the zero-sum rule can be independently justified. (Of course it follows trivially from Eq.(2), but this was derived from Eq.(1), and at this point we cannot make use of the concept of probability.) Here is an argument: banking too is a form of gambling; the only difference between acting as the gambler who bets, and as the banker who accepts the bet, is that whereas the gambler *pays* a stake in order to play, and *receives* payoffs according to the outcomes, the banker *receives* the stake in order to play, and *pays* the payoffs according to the outcomes. The zero-sum rule is the statement that the most that one will pay in the hope of gaining a utility is the least that one will accept to take the risk of losing it. We may take it that *this* principle, as a principle of zero-sum games, is perfectly secure. And evidently any quantum experiment can be used to play a zero-sum game; therefore this principle also applies to the expected utility of experiments.

What of the general equal-norm case, Case 2? Here the zero-sum rule is not enough. But if we consider only the case $d = 2$, it is enough to supplement it with another rule, what Deutsch calls the *additivity* rule. A payoff function $\Omega : R \rightarrow U$ is *additive* if and only if $\Omega(x + y) = \Omega(x) + \Omega(y)$. Let $f_k : R \rightarrow R$ be the function $f_k(x) = x + k$; then V is *additive* if and only if

$$V[\psi, \hat{X}, \Omega \circ f_k] = V[\psi, \hat{X}, \Omega] + \Omega(k). \quad (26)$$

Additivity of the payoff function is a standard assumption of elementary decision theory, eminently valid for small bets (but hardly valid for large ones, or for utilities that only work in tandem). Additivity of V then has a clear rational: it is an example of a *sure-thing principle*, that if, given two games, each exactly the same, except that in one of them one receives an additional utility $\Omega(k)$ *whatever* the outcome, then one should value that game as having an *additional* utility $\Omega(k)$.

To see how additivity can be used in Case 2 (but restricted to $d = 2$), observe that for $k = -\lambda_1 - \lambda_2$, the function $-I \circ f_k$ is the permutation π . Therefore from Eq.(14) we may conclude:

$$V[\psi, \hat{X}, \Omega] = V[\psi, \hat{X}, \Omega \circ -I \circ f_k]. \quad (27)$$

By additivity the RHS is $V[\psi, \hat{X}, \Omega \circ -I] + (\Omega \circ -I)(k)$, and since Ω is additive (so $\Omega \circ -I = -\Omega$) we obtain, from the zero-sum rule

$$V[\psi, \hat{X}, \Omega] = -V[\psi, \hat{X}, \Omega] - \Omega(k). \quad (28)$$

With a further application of payoff additivity there follows

$$V[\psi, \hat{X}, \Omega] = \frac{1}{2}[\Omega(\lambda_1) + \Omega(\lambda_2)] \quad (29)$$

in accordance with the Born rule.

As Wallace has shown, this, along with the higher dimensional cases ($d > 2$), can be derived from much weaker axioms of decision theory, that do not assume additivity. Theorem 5 then goes through unchanged.⁹ As already remarked, one is then in a position to derive the extension to the irrational case without assuming continuity: for the details, I refer to Wallace [12].

Decision theory can evidently play a role in the derivation of the Born rule, but it is only needed if the notion of probability is itself in need of justification. That may well be so, in the context of the Everett interpretation; but on other approaches to quantum mechanics, probability, whatever it is, can be taken as given.

12 Gleason's Theorem

Compare Gleason's theorem:

Theorem 8 *Let f be any function from 1-dimensional projections on a Hilbert space of dimension $d > 2$ to the unit interval, such that for each resolution of the identity $\{\hat{P}_k\}$, $k = 1, \dots, d$, $\sum_{k=1}^d \hat{P}_k = I$, $\sum_{k=1}^d f(\hat{P}_k) = 1$. Then there exists a unique density matrix ρ such that $f(\hat{P}_k) = \text{Tr}(\rho \hat{P}_k)$.*

Proof. Gleason (1967) ■

A first point is that the derivation of the Born rule presented here concerns the notion of a *fixed* algorithm that applies to arbitrary measurement models, hence to Hilbert spaces of *arbitrary* dimension, whereas Gleason's theorem concerns an algorithm that applies to *arbitrary* resolutions of the identity on a Hilbert space of *fixed* dimension. Although the proof of Theorems 5 and 7 made

⁹It is worth remarking that a derivation of the Born rule for initial states that are not normalized to unity is just what is needed for the Everett interpretation, as also the de Broglie-Bohm theory (in reality, according to either approach, one *always* deals with branch amplitudes with modulus strictly less than one - supposing the initial state of the universe has modulus one).

use of a Hilbert space of large dimensionality, it applies to the 2-dimensional case as well.

More important, on a variety of approaches to quantum mechanics, nothing so strong as Gleason's premise is really motivated. It is not required that probabilities can be defined for a projector independent of the family of projectors of which it is a member. This requirement, sometimes called *non-contextuality* [8], is very strong. Very few approaches to quantum mechanics subscribe to it. The theorem has no relevance to any approach that singles out a unique basis once and for all: it applies neither to the GRW theory [5], nor to the de Broglie-Bohm theory [6], which single out the position basis; it does not apply to the Everett interpretation [?], which singles out a basis approximately localized in phase space; it does not apply to the consistent histories approach [3], assuming the choice of decoherent history space is unique. All these theories require *only* that probabilities be defined for projectors associated with the preferred basis - if they apply to any other resolution of the identity, it is insofar as in a *particular* context, experimental or otherwise, the latter projections become correlated with members of the former family.

But so much is entirely compatible with the derivation that we have offered. By all means restrict Definition 1 to observables compatible with a unique resolution of the identity (and likewise the consistency condition of Definition 2). Lemma 3 proves identities for expectation values for commuting observables, it likewise can be restricted to a unique resolution of the identity; likewise Theorem 4. In Theorem 5 an auxiliary basis was used, but again this can again be taken as the preferred basis. And whilst it is in the spirit of Theorem 7 that probabilities should also be defined for small variations in projectors, this does not yet amount to the assumption of non-contextuality.

Unlike the premise of Gleason's theorem, the operational criteria that we have used are hardly disputed; they are common ground to all the major schools of foundations of quantum mechanics. But it would be wrong to suggest that they apply to all of them equally: on some approaches - in particular, those that provide a detailed dynamical model of measurements - there is good reason to suppose that an algorithm for expectation values will depend on additional factors (in particular, on the state at the instant of state reduction); the Born rule may no longer be forced in consequence. (But we take it that this would be an *unwelcome* consequence of these approaches; the Born rule will have to be *otherwise* justified - presumably, as it is in the GRW theory, as a *hypothesis*).

Of the major schools, two - the Everett interpretation, and those based on operational assumptions (here we include the Copenhagen interpretation) - offer no such resources. This point is clear enough in the latter case; in the case of the Everett interpretation, the association of models with multiple channel experiments as given in Definition 1 follows from the full theory of measurement.¹⁰ Quantum mechanics under the Everett interpretation provides no leeway in this matter. The same is likely to be true of any approach to

¹⁰For arguments in the still more general case, on applying the Everett theory of measurement to any experiment, I refer to Wallace [12] (see in particular his principle of "measurement neutrality").

quantum mechanics that preserves the unitary formalism intact, without any supplement to it.

The principal remaining schools have a rather different status. One, the state-reduction approach, has already been remarked on: a new and detailed dynamical theory of measurement is likely to offer novel definitions of experimental models and novel criteria for when they are to be applied. The other is the hidden-variable approach, in which the state evolves unitarily even during measurements (but is incomplete). This case deserves special consideration.

13 Completeness

As it happens, the one approach to foundations in which the Born rule has been seriously questioned is an example of this type (the de Broglie-Bohm theory)[11]. Hidden variables certainly make a difference to the argument we have presented. Consider the proof of Theorem 4. The passage from Eq.(13) to Eq.(14) hinged on the fact that the state on both sides of Eq.(13) is *identical* when the norms of its components are the same. (Likewise the step from Eq.(10) to (11).) But if the state is incomplete, this is not enough to ensure the required identification. Including the state of the hidden variables as well (denote w), we should replace ψ by the pair $\langle \psi, \omega \rangle$ (w may be the value of the hidden variable, or a probability distribution over its values). Doing this, as Wallace has pointed out [12], there is no guarantee that in the case of superpositions of equal norms - e.g. for $\psi = \frac{1}{\sqrt{2}}(\varphi_1 + \varphi_2)$, where φ_1, φ_2 are, as in Case 1, eigenstates of the z -component of spin - that $\hat{U}_\pi$ (permuting φ_1 and φ_2) will act as the identity. Although $\hat{U}_\pi \psi = \psi$, its action on $\langle \psi, \omega \rangle$ may well be different from the identity; how is the permutation to act on the hidden variables?

The question is clearer when $\hat{U}_\pi$ implements a spatial transformation. We have an example where it does: the Stern-Gerlach experiment. In this case $\hat{U}_\pi \hat{\sigma}_z \hat{U}_\pi^{-1} = -\hat{\sigma}_z$, a reflection in the $x - y$ plane. Under the latter, a particle initially with positive z -coordinate ($\omega = +$) is mapped to one with negative z -coordinate ($\omega = -$). Under this same transformation, the superposition $\psi = \frac{1}{\sqrt{2}}(\varphi_1 + \varphi_2)$ is unchanged. Therefore $\hat{U}_\pi : \langle \psi, + \rangle \rightarrow \langle \psi, - \rangle \neq \langle \psi, + \rangle$; there is no longer any reason to suppose that Eq.(11) will be satisfied.

This situation is entirely as expected. In the de Broglie-Bohm theory, given such an initial state ψ , it is well known that if the incident particle is located on one side of the plane of symmetry of the Stern-Gerlach apparatus, then it will always remain there. It is obvious that if the particles is always located on the same side of this plane, on repetition of the experiment, the statistics of the outcomes will disagree with the Born rule. It is equally clear that if particles are randomly distributed about this plane of symmetry then the Born rule will be obeyed - but that is only to say that the probability distribution for the hidden variables is determined by the state, in accordance with the Born rule. This is what we are trying to prove.

But it does not follow that the arguments we have given have no bearing on

such a theory. Our strategy, recall, was to derive constraints on an algorithm - any algorithm - that takes as inputs experimental models and yields as outputs expectation values. The constraints will apply even if the state is incomplete, even if there are additional parameters controlling individual measurement outcomes - so long as the state *alone* determines the statistical distribution of the hidden variables. Given that, then any symmetries of the state will also be symmetries of the distribution of hidden variables. In application to the de Broglie-Bohm theory, our result indeed implies that the particle distribution *must* be given by the Born rule - this is no longer an additional postulate of the theory - *so long as the particle distribution is determined only by the state*. The assumption is not that particles must be distributed in accordance with the Born rule, but that they are distributed by any rule at all that is determined by the state. Then it is the Born rule.

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